Graduate Program in Economics, Binghamton University

ECON 503: Mathematic Analysis for Economists

Diagnostic Exam

Question 1

a) With reference to the function $f : R \rightarrow R$,

   i. State the Mean Value Theorem of the differential calculus (MVT);
   ii. State the Taylor Series Expansion of $f(x)$ about the point $x = a$

   Carefully demonstrate that the MVT is a special case of the Taylor Series.

b) Use the Taylor Series to prove that $f'(a) = 0$ is a necessary condition for the function f to have a relative extrema at $(a, f(a))$. Furthermore, prove that if $f'(a) = f''(a) = 0$ and $f'''(a) \neq 0$, then the point $(a, f(a))$ is neither a relative minimum nor a relative maximum point.

c) By considering the Taylor Series expansion of $f(x) = e^x$ about $x = 0$, show that

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + R$$

   Where $R < \frac{1}{8}$

Question 2

Suppose that aggregate consumption in period t, C_t is a linear function of aggregate income in the previous period, Y_t, that is,

$$C_t = A + BY_{t-1}$$

where A and B are positive constants. Let aggregate investment be a constant amount I and aggregate income be equal to consumption plus investment, that is,

$$Y_t = C_t + I$$

a) Derive the difference equation for aggregate income Y_t.
b) Solve the difference equation you obtained in part (i)
c) Find the steady state value of Y.
d) What restrictions must be placed on B to ensure that income converges monotonically to the steady state equilibrium?

Question 3

Consider a representative agent consumes two goods, economic books and bread. Her utility function is given by $U(X,Y) = X^{\alpha}Y^{\beta}$, where X stands for the number of economic books and Y stand for the number of bread. α and β are such that $\alpha + \beta = 1$.

a) Derive her demand for economic books and her demand for bread.

b) Define $s_i = \frac{P_i X_i}{I}$ as the income share of good *i*, and η_i as the income elasticity of the good *i*. Verify that identity that $s_X \eta_X + s_Y \eta_Y = 1$

c) Is your result in (2) invariant with respect to the form of the utility function? Prove your result.

Question 4

An optimal control model is represented by the following:

$$\text{Maximize} \quad \int_0^2 (y - u^2)\,dt$$

$$\text{Subject to} \quad \dot{y} = u$$

$$\text{And} \quad y(0) = 0 \qquad y(2) \text{ free} \qquad u(t) \text{ unconstrained}$$

Using the Maximum Principle

a) Write down the Hamiltonian function for this problem.

b) Find the optimal paths for the control, state, and co state variables.

c) Show that your optimal path for the control variable maximizes the Hamiltonian.

d) Sketch the maximum path for the control variable.

Question 5

In the study of econometrics, we sometimes encounter an autoregressive process, which means that a variable y is regressed upon itself. An example of such a process can be expressed by the difference equation

$$y_t = \beta y_{t-1} + e_t, \quad \beta < 1, \qquad (*)$$

where e is a random error term having the form

$$e_t = \rho e_{t-1} + u_t$$

with u_t also being an error term.

a) By substituting for e_t in equation (*) and then eliminating e_{t-1} from the resulting equation, show that

$$y_{t+2} - (\beta + \rho) y_{t+1} + \rho\beta y_t = u_{t+2}$$

b) Given that $\rho = \beta = -\frac{1}{2}$ and $u_t = 0$, solve the difference equation that you obtained in part (a).

Question 6

Given the following non-linear maximization problem:

$$\text{Maximize} \quad Z = 2x_1^2 + 8x_2^2$$

$$\text{Subject to} \quad \begin{array}{l} x_1^2 + x_2^2 \le 16 \\ x_1, x_2 \ge 0 \end{array}$$

a) Formulate the Lagrangian function for this problem.

b) Derive the Kuhn-Tucker conditions.

c) What is the optimal solution to this problem?

Question 7

a) A quadratic form is given by $Q(x, y, z) = 2x^2 + 3y^2 - 4xy$. By completing the square, express $Q(x,y,z)$ as a sum of squares. Is Q positive definite? Give reasons.

b) Let $Q = x^T Ax$, where A is a symmetric matrix, be a given quadratic form. Q is transformed to a new quadratic form Q^1 by a linear transformation $x = Py$. Show that the square matrix corresponding to Q^1 is $P^T AP$ and demonstrate that it is symmetric.